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Notes on the Macroeconomic Determinants of Mean Reversion†

John B. Donaldson
Columbia University

Natalia Gershun
Pace University

Rajnish Mehra
Arizona State University
Luxembourg School of Finance and NBER

Abstract
We reflect on the property of mean reversion in stock prices and returns within a class of dynamic stochastic general equilibrium macroeconomic models. Our objective is to understand the macroeconomic structures responsible for mean reversion and to gain insight regarding the observed difficulty in detecting mean reversion in actual return and price data.

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1. Introduction

Beginning with Cochrane (2010) there has been renewed interest in the phenomenon of equity return predictability. Since a predictable return series in the sense of Cochrane (2010) must also be mean reverting, this latter concept has also moved center stage.[1]

A review of the literature, however, reveals that there are several operational definitions of mean reversion. Accordingly, in the first portion of the paper we examine these various characterizations of mean reversion and check for consistency among them.

We then go on to explore the pervasiveness of mean reversion in equity returns within the context of a series of increasingly complex macroeconomic models. More specifically, and recognizing that traded equity securities represent ownership of a substantial portion of a nation's capital stock, we explore the mean reverting property of equity returns and the equity premium within the context of standard DSGE macroeconomic models parameterized to replicate the patterns in macroeconomic time series found in United States macroeconomic data. Is mean reversion in equilibrium equity returns a standard characteristic of this model class? What are the model features principally responsible? It is these questions that we explore.

We reach a number of conclusions:

1. We conclude that standard characterizations of mean reversion are limited in the precision of the information they propose to convey.[2] We suggest an alternative characterization of "mean reversion," one that provides not only a more intuitive measure of the degree of "mean reversion," but also provides a simple sense of one series being "more strongly mean-reverting" than another.

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[1] Cochrane (2011) first considers "predictability" regressions of the form $\tilde{r}^{p}_{t,t+k} = a + b(d_t/p^e_t) + \tilde{\varepsilon}_{t+k}$ where $r^{p}_{t,\,t+k}$ is the cumulative excess annual return on the market index from t to $t+k$, and d_t / p^e_t is the dividend price ratio at t. He finds $b > 0$ and statistically significant when $k = 1, 5$ years.

In this paper we will consider more immediate regressions of the form $\tilde{r}_{t,t+k} = a + b_k\tilde{r}_{t-k,t} + \tilde{\varepsilon}_{t+k}$, where a statistically significant $b_k < 0$ is deemed to be evidence of predictability. If $b_k < 0$, then clearly $corr(\tilde{r}_{t-k,t}, \tilde{r}_{t,t+k}) < 0$, which is a frequent characterization of "mean reversion," hence, the close association of these concepts. In this latter calculations the parameter "k" identifies the "time horizon" of the mean reversion.

[2] In fact, using customary measures, we argue the mean reversion/aversion distinction is entirely arbitrary and therefore meaningless.

2. For simple production-based dynamic asset pricing models and using customary characterizations, we find that mean reversion in equity returns or the premium, as equilibrium phenomena, is the exception rather than the rule: mean aversion is the more generic property (we precisely define these terms in the next section).[3] This result makes less puzzling the difficulty in recovering strong proof of equity return mean reversion in the data: as a competitive equilibrium phenomenon we have no theoretical reason to expect its existence, at least as customarily characterized.[4] In particular, the introduction of persistence in total factor productivity shocks, increasing the degree of representative investor risk aversion, the introduction of habit formation preferences, the introduction of a competitive labor market, or the inclusion of a cost of adjusting the level of the capital stock are all features that tend to work against mean reversion as it is typically defined.[5]

3. In contrast, we do find evidence of mild mean reversion in excess returns in models where the underlying source of uncertainty takes the form of variation in the share of income to capital rather than variation in economy-wide total factor productivity.

A fair summary of the empirical literature suggests conflicting empirical evidence for mean reversion in stock prices and returns. Summers (1986), Campbell and Mankiw (1978), Fama and French (1988), Lo and MacKinley (1988) and Poterba and Summers (1988) report mean reversion in stock returns over time horizons (see footnote 1) less than ten years. Poterba and Summers (1988), for example, find positive equity return autocorrelation over horizons of less than one year and negative autocorrelation over longer periods (though not significant at the 5% level). The conclusions to these studies have subsequently been placed in doubt on purely statistical grounds. Kim, Nelson and Startz (1991), in particular, demonstrate that due to small sample bias, the conclusions reached in Fama and French (1988b) and Poterba and Summers (1988) are invalid. In fact, Richardson and Stock (1989) and Richardson (1993)

[3] DSGE abbreviates "dynamic stochastic general equilibrium.

[4] This assertion holds for models where the time interval is calibrated to be a quarter or a year. We have nothing to say about daily return patterns which may well be governed by very different mechanisms.

[5] In doing so we extend the work of Basu and Vinod (1994) to richer macroeconomic settings.

suggest that a proper accounting for small sample bias reverses the conclusions in these latter two papers.[6]

The principal challenge facing this literature is the absence of adequate data for discriminating statistical tests. As with Cecchetti et al. (1990), the modeling environment we consider sidesteps the paucity of actual return data by allowing us to generate arbitrarily long return series, and thus undertake our statistical analysis with a high degree of precision. Nevertheless, this particular modeling feature is of limited significance if the empirical quantities to be matched are not themselves known with a high degree of precision.

The primary intellectual antecedents of the present study are Basu and Vinod (1994), Cecchetti et al. (1990), Guvenen (2009), and Lansing (2015). In a Lucas (1978) style exchange model where dividends follow a Markov switching regime (see Hamilton (1989)), Cecchetti et al. (1990) were then able to replicate the observed patterns of mean reversion measures (variance ratios and β-regression coefficients) at various horizons. We undertake some of the same measurements in models where the dividend series arise endogenously as the result of consumption and investment decisions by households and firms. So do Guvenen (2009) and Lansing (2015), although equity return mean reversion is a minor focus of these works. Guvenen (2009) explores asset pricing in a model where firm owners and workers have differential access to securities markets: firm owners trade both equity and default free bonds while workers are limited to bond trading. Lansing (2015) explores the asset pricing consequences of variation in factor shares. Both report slight negative correlation in equity returns based on data, and as equilibrium outcomes of their models. Since simple DSGE models give strongly positive autocorrelation in equity returns, another objective of the present paper is to ascertain the distinct features responsible for this "transformation" relative to simpler antecedent models. Lastly, we note that the analysis in Basu and Vinod (1994) initiated some of the same explorations and serves to motivate the present study.

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[6] Similar confusion reigns in the literature concerning predictability. See the claims in Fama (1981), Campbell (1987), Fama and French (1988a, b), Campbell and Shiller (1988) and Fama and French (1989), and their refutation on statistical grounds in Richardson and Stock (1989), Nelson and Kim (1993), Cavanaugh et al. (1995) and Stambaugh (1999). See also the conflicting perspectives in, e.g., Lewellen (2004), Torous and Valkanor (2007), and Campbell and Yogo (2006) versus Goyal and Welch (2003, 2008) and Bossaerts and Hillion (1999).

An excellent summary of the literature can be found in Zakamulin (2015), which explores the evidence for mean reversion/predictability in periods exceeding ten years.

An outline of the paper is as follows. In section 2 we identify three definitions of mean reversion found in the literature, and partially characterize their interrelationships. In Section 3, these definitions are then applied to the analysis of a basic Benchmark dynamic macroeconomic model. An alternative characterization of "mean reversion" is proposed. In Section 4 we add additional features to the benchmark model and study the resulting implications for the strength of mean reversion/aversion in model-generated equity return and equity premium data. Section 5 provides a behavioral perspective on mean reversion. Section 6 concludes.

Our analysis is both analytical and, as frequently necessary, computational as based on wide ranging numerical simulations.

2. Mean Reversion

There is no unique property attached to the expression "mean reversion." When applied to an economic time series, the intuitive notion that the expression "mean reverting" conveys is that of a time series which periodically assumes values above and below its mean, transitioning from one set of values to the other at fairly regular intervals. Mean reversion has been historically identified with the concept of stationarity, but this is surely inadequate since any i.i.d. process is stationary but not necessarily mean-reverting in any discriminating sense of that word. Within the finance literature there appear to be three distinct candidate properties identified with "mean reversion." They are as follows, expressed in terms of an arbitrary stationary stochastic process $\{\tilde{x}_t\}$.

A stationary stochastic process $\{\tilde{x}_t\}$ is said to be mean reverting if and only if:

I. $cov(\tilde{x}_t, \tilde{x}_{t+1}) < 0$ (1)[7]

[7] More generally, the process $\{\tilde{x}_t\}$ is said to be mean-reverting at a lag of *j* periods if and only if $cov(\tilde{x}_{t-j}, \tilde{x}_t) < 0$. These notions are equivalent to a negative $\hat{\beta}_j$ coefficient in the regression
$\tilde{x}_t = \hat{\alpha}_j + \beta\tilde{x}_{t-j} + \tilde{\varepsilon}_{t,j}, j = 1,2,$ .... etc.
Other authors focus on cumulative quantities, in particular, returns (e.g., Cecchetti et al. (1990)) as per

$$\text{cov}\left(X_{t-j,t}, X_{t,t+j}\right) < 0,$$

where $X_{t-j,t} = \sum_{i=0}^{j-1} x_{t-i}$ and $X_{t,t+j} = \sum_{i=1}^{j} x_{t+i}$.
The regression counterpart to the latter case is
$\tilde{X}_{t,t+j} = \hat{\alpha}_j + \beta\tilde{X}_{t-j,t} + \tilde{\varepsilon}_{t,t+j}$,
with $\hat{\beta}_j < 0.$ We elect to focus on the simplest representation (1).

We will demonstrate that this characterization is virtually meaningless in the sense that most model generated financial time series do not satisfy it. Yet, they do respect the basic intuition underlying "mean reversion" mentioned above.

II. $\frac{var(\tilde{x}_t+\tilde{x}_{t+1}+\cdots+\tilde{x}_{t+j})}{j+1} < var(\tilde{x}_t)$, for any $j \geq 1$. (2)

Property I is used by Guvenen (2009) and Lansing (2014). Property II was first proposed in Summers (1986) and used in, e.g., Poterba and Summers (1988) and Mukherji (2011) for their discussions of mean reversion in stock price and rate of return series.[8]

III. for any time integers $0 \leq r < s < t < u$,

$cov(x_s - x_r, x_u - x_t) < 0$. (3)

Property III, to our knowledge first proposed in Exley et al. (2009), is a comment about sequential *changes* in the values of the stochastic process $\{\tilde{x}_t\}$ rather than a statement about the statistical properties of the values themselves. Interpreting$\{\tilde{x}_t\} = \{\tilde{k}_t\} = \{\tilde{p}_t^e\}$, Property III suggests that increases in the price of capital over a particular interval of time will generally be followed by reductions in the price in future time intervals. As such, it represents a different sense of mean reversion than Properties I and II. In each of the definitions if the identifying inequality is reversed, the series is said to be mean averting.[9]

The relationship between Properties I and II is partially captured in Proposition 2.1.

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[8] It stands in specific contrast to the analogous property of a random walk where $var(\tilde{x}_t + \tilde{x}_{t+1} + \cdots + \tilde{x}_{t+j-1}) = jvar(\tilde{x}_t)$.

[9] Property III can be guaranteed if certain sufficient conditions are satisfied. This is the subject of the following Lemma and Proposition:

Lemma 2.1: Consider arbitrary time indices $0 < r < s < t < u$. A stochastic process $\{\tilde{x}_t\}$ is mean reverting by Property III if and only if

$var(\tilde{x}_u - \tilde{x}_r) + var(\tilde{x}_t - \tilde{x}_s) < var(\tilde{x}_t - \tilde{x}_r) + var(\tilde{x}_u - \tilde{x}_s)$. (5)

Proof: See Exley et al. (2004).

Let us next make the identification

$v(h-k) \equiv_{def} var(\tilde{x}_h - \tilde{x}_k)$ for any time integers $h > 0, k > 0$. This allows a simple presentation of the following proposition:

Proposition 2.2: If $v(\quad)$ is concave then $\{\tilde{x}_t\}$ is mean reverting by Property III.

Proof: See Exley et al. (2004).

Proposition 2.1 Let $\{\tilde{x}_t\}$ be a stationary stochastic process with an ergodic probability distribution. With respect to that distribution, statistical properties I and II detailed above are related according to

a) II ➔ I

b) If $|cov(\tilde{x}_t, \tilde{x}_{t+1})| > \left|\sum_{s=2}^{j-1} cov(\tilde{x}_t, \tilde{x}_{t+s})\right|$ for all j (4)

then I ➔ II.

Proof: See the Appendix.

As regards the empirical literature on mean reversion, Property III is rarely employed and neither implies nor is implied by either Property I or II. Accordingly, we largely focus principally on characterizations I and II; in particular, are they satisfied in equilibrium macroeonomic models?

3. Modeling Perspective and the Benchmark Paradigm

Our initial focus will be macroeconomic models for which the fundamental underlying structure is the one good stochastic growth model with "planning" representation:

$$maxE(\textstyle\sum_{t=0}^{\infty} \beta^t u(\tilde{c}_t, 1 - \tilde{n}_t)) \quad (6)$$

s.t. $c_t + i_t \leq y_t = f(k_t, n_t)\tilde{\lambda}_t$

$k_{t+1} = (1 - \Omega)k_t + i_t$, k_0 given.

$\tilde{\lambda}_{t+1} \sim G(\tilde{\lambda}_{t+1}, \tilde{\lambda}_t)$.

Adopting the customary notation, $u(c_t, 1 - n_t)$ represents the representative agent's period utility function defined over his period t consumption c_t and leisure, $(1 - n_t)$, where n_t is labor supplied, $f(k_t, n_t)\lambda_t$ denotes the representative firm's CRS production function of capital stock k_t and labor supplied with $\{\tilde{\lambda}_t\}$ the stochastic total factor productivity shock. The probability distribution function for $\{\tilde{\lambda}_{t+1}\}$ conditional on λ_t is denoted $G(\tilde{\lambda}_{t+1}, \tilde{\lambda}_t)$ and is assumed known to the representative agent.[10] Lastly, b denotes the representative agent's subjective time discount factor and W the period depreciation rate.

[10] The productivity disturbance $\{\tilde{\lambda}_t\}$ will typically be of the form $\tilde{\lambda}_t = e^{\tilde{x}_t}$ where $\tilde{x}_t$ is an A.R.-1 process.

As the previous notation suggests, the state variables for this economy are k_t and l_t. Under standard assumptions, problem (6) has a solution; that is:

(i) continuous, time-invariant consumption $c_t = c(k_t, \lambda_t)$, investment $i_t = i(k_t, \lambda_t)$ and labor service $n_t = n(k_t, \lambda_t)$ functions exist that solve problem (6), and

(ii) a unique invariant probability measure on the state variable pair (k_t, λ_t) exists to which the joint stochastic process on (k_t, λ_t) converges weakly and which describes its long run behavior. With these attributes we say that the joint process on (k_t, λ_t) is stationary.[11, 12] As a result the stochastic processes governing investment, i(k_t, λ_t), consumption, c(k_t, λ_t), labor service, n(k_t, λ_t) and output, $y_t = y\big(k_t, n(k_t, \lambda_t)\big)\lambda_t$ are also stationary. The same investment and consumption functions arising as the solution to (6) coincide with the aggregate investment and consumption functions arising from an analogous decentralized market economy in recursive competitive equilibrium, a fact well known to the literature; see, e.g., Prescott and Mehra (1980), Brock (1982), or Danthine and Donaldson (2015).

These decentralization schemes for (6) may be generalized to accommodate an implied financial market where risk free debt and equity are competitively traded.[13] Under this expanded interpretation the period t dividend satisfies

$$d_t = f(k_t, n_t)\lambda_t - w_t n_t - i_t \quad (7i)$$

[14]

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[11] Our notion of stationarity for a discrete time Markov process $\{x_t\}$ is as follows; Let s, t be arbitrary time indices and X the state space with $\hat{x} \in X$, and $B \subseteq X$, B a subset.
Define $P\left(s, \hat{x}, t, B\right) = \text{Prob}\left(x_t = B; x_s = \hat{x}\right)$. Then for any integer u, if $P\left(s+u, \hat{x}, t+u, B\right) = \text{Prob}\left(x_{s+u} = \hat{x}; x_{t+u} \in B\right) = P\left(s, \hat{x}, t, B\right)$, the Markov process is said to be stationary. The same Markov process possesses an invariant distribution $\hat{G}(\;)$ on X if and only if for any $B \subseteq X$,

$$\text{Prob}\left(x_{t+1} \in B\right) = \int_{x \in X} P\left(x_{t+1} \in B;\, x_t = x\right)\hat{G}\left(dx\right).$$

All the stochastic processes analyzed in this article are Markov, stationary and possess unique invariant distributions defined on compact sets.

[12] The details behind these assertions can be found in the literature. Part (i) is entirely standard. As for part (ii), the stochastic kernel, the expression $P\left(s, \hat{x}, t, B\right)$ in footnote 3 can be shown to be increasing, order reversing and to satisfy the "Feller Property." By Theorem 3.2 in Kamihigashi and Stachurski (2014) a unique stationary probability distribution exists with the indicated properties.

[13] As such, the financial market can be regarded as "complete."

[14] In a related study, Lansing (2015) refers to this dividend expression as the "macroeconomic dividend."

while the ex dividend aggregate equity price, $p_t^e = p^e(k_t, \lambda_t)$, is identified with next period's capital stock:

$$p_t^e = k_{t+1}. \tag{7ii}$$

In (7i) w_t denotes the competitive wage rate which, in equilibrium, satisfies

$$w_t = f_2(k_t, n_t)\lambda_t.$$

Accordingly,

$$1 + r_{t+1}^e = \frac{p_{t+1}^e + d_{t+1}}{p_t^e} \tag{8}$$

$$= \frac{k_{t+2} + f(k_{t+1}, n_{t+1})\lambda_{t+1} - n_{t+1} f_2(k_{t+1}, n_{t+1})\lambda_{t+1} - i_{t+1}}{k_{t+1}}$$

$$= \frac{k_{t+2} + k_{t+1} f_1(k_{t+1}, n_{t+1})\lambda_{t+1} - i_{t+1}}{k_{t+1}} \qquad \text{(by CRS)}$$

$$= f_1(k_{t+1}, n_{t+1})\lambda_{t+1} + 1 - \Omega \tag{9}$$

where $r_{t+1}^e = f_1(k_{t+1}, n_{t+1})\lambda_{t+1} - \Omega$ denotes the net return on unlevered equity from the "end of period t" to the "end of period t+1." [15]

The period price, p_t^b, of a risk-free bond paying one unit of consumption in period t+1, irrespective of the realized state, is

$$p_t^b = p^b(k_t, \lambda_t) = \beta \int \frac{u_1(\tilde{c}_{t+1}, 1 - \tilde{n}_{t+1})}{u_1(c_t, 1 - n_t)} dG(\tilde{\lambda}_{t+1}; \tilde{\lambda}_t) \tag{10}$$

with the risk-free rate $r_t^b = r^b(k_{t-1}, \lambda_{t-1})$ satisfying $(1 + r_{t+1}^b) = 1/p_t^b$. Accordingly, the equity premium is defined by $r_t^p = r_t^e - r_t^b$.

As continuous bounded functions of the economy's state variables, $p^e(k_t, \lambda_t)$, $p^b(k_t, \lambda_t)$, $r^e(k_t, \lambda_t)$ and $r^b(k_t, \lambda_t)$ are also stationary stochastic processes. In addition, the capital-output ratio, $\left\{\frac{k_t}{y_t}\right\}$, the growth rate of output, $\{g_t^y\} = \left\{\frac{y_t}{y_{t-1}}\right\}$, and the share of income to capital, $\left\{\frac{r_t^e k_t}{y_t}\right\}$, represent the ratios of strictly positive stationary stochastic processes and thus are stationary as well. These latter quantities will be relevant for our discussion to follow.

[15] Identification (9) does not necessarily hold in more elaborate models; see Section 4.

3.1. The Benchmark Model

We restrict problem (6) by requiring that $u(c_t) = \ell n(c_t)$ while $y_t = f(k_t, n_t)\tilde{\lambda}_t = \tilde{k}_t^{\alpha}\tilde{\lambda}_t$ where labor n_t is fixed at $n_t \equiv 1, \Omega = 1$, and $\{\tilde{\lambda}_t\}$ is a strictly positive i.i.d. stochastic process. It is widely known that the optimal policy functions assume the form

$$c_t = c(k_t, \lambda_t) = (1 - \alpha\beta)y_t \text{ and} \tag{11}$$

$$i_t = k_{t+1} = \alpha\beta y_t = \alpha\beta k_t^{\alpha}\lambda_t, \text{ while} \tag{12}$$

$$d_t = \alpha k_t^{\alpha}\lambda_t - \alpha\beta k_t^{\alpha}\lambda_t = \alpha(1-\beta)k_t^{\alpha}\lambda_t.\text{[16]} \tag{13}$$

Accordingly,

$$p_t^e = k_{t+1} = p^e(k_t, \lambda_t) = \alpha\beta k_t^{\alpha}\lambda_t.$$

It can also be shown that

$$p_t^b = p^b(k_t, \lambda_t) = \left(\frac{\beta E(\lambda_t^{-1})}{(\alpha\beta)^{\alpha}}\right) k_t^{\alpha(1-\alpha)}\lambda_t^{1-\alpha}. \tag{14}$$

With $\{\tilde{\lambda}_t\}$ an i.i.d. process, Danthine and Donaldson (1981) and Hopenhayn and Prescott (1992) have shown that the derived stochastic process on capital stock is stationary and that there exists a corresponding ergodic probability distribution which captures its long run behavior; the same can thus be said for p_t^e and p_t^b.

Let us first explore this model as regards Property I.

<u>Proposition 3.1</u>: Consider Model (1) specialized as per (11) and (12). The equity price, the equity dividend series and the risk free bond price series are all mean-averting by Property I.[17]

Proof:

a. The equity price relationship follows from a double application of Jensen's inequality:

$$cov(\tilde{p}_{t-1}^e, \tilde{p}_t^e) = cov(\tilde{k}_t, \tilde{k}_{t+1}) = cov(\tilde{k}_t, \alpha\beta\tilde{k}_t^{\alpha}\tilde{\lambda}_t)$$

$$= \alpha\beta E(\tilde{\lambda}_t)\{E(\tilde{k}_t^{1+\alpha}) - E(\tilde{k}_t)E(\tilde{k}_t^{\alpha})\}$$

$$> \alpha\beta E(\tilde{\lambda}_t)\left\{E(\tilde{k}_t^{1+\alpha}) - E(\tilde{k}_t)\left(E(\tilde{k}_t)\right)^{\alpha}\right\}$$

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[16] This identification of the dividend assumes that investment comes out of capital's share. By recursive substitution $k_t = \left[(\alpha\beta)^{1+\alpha+\alpha^2+\cdots+\alpha^{t-1}}\right]k_0^{\alpha^t}\prod_{s=0}^{t-1}\lambda_s^{\alpha^{t-1-s}}$.

[17] Part a of Proposition 3.1 was first presented in Basu and Vinod (1994) and Basu and Samanta (2001) in a slightly less general setting. We extend their explorations with a different goal in mind.

$$= \alpha\beta E(\tilde{\lambda}_t)\left\{E(\tilde{k}_t^{1+\alpha}) - \left(E(\tilde{k}_t)\right)^{1+\alpha}\right\} > 0.$$

b. By a similar derivation (see the Technical Appendix), $cov(d_t, d_{t+1}) > 0$. ■

c. From

$$p_t^b = \frac{\beta E(\lambda_t^{-1})}{(\alpha\beta)^\alpha} k_t^{\alpha(1-\alpha)} \lambda_t^{1-\alpha},$$

$$cov(p_t^b, p_{t+1}^b) = cov\left(\left(\frac{\beta E(\lambda^{-1})}{(\alpha\beta)^\alpha}\right)\tilde{k}_t^{\alpha(1-\alpha)}\tilde{\lambda}_t^{1-\alpha}, \left(\frac{\beta E(\lambda^{-1})}{(\alpha\beta)^\alpha}\right)\tilde{k}_{t+1}^{\alpha(1-\alpha)}\tilde{\lambda}_{t+1}^{1-\alpha}\right)$$

$$= \left(\frac{\beta E(\lambda^{-1})}{(\alpha\beta)^\alpha}\right)^2 cov\left(\tilde{k}_t^{\alpha(1-\alpha)}\tilde{\lambda}_t^{1-\alpha}, \tilde{k}_{t+1}^{\alpha(1-\alpha)}\tilde{\lambda}_{t+1}^{1-\alpha}\right)$$

$$= \left(\frac{\beta E(\lambda^{-1})}{(\alpha\beta)^\alpha}\right)^2 cov\left(\tilde{k}_t^{\alpha-\alpha^2}\tilde{\lambda}_t^{1-\alpha}, (\alpha\beta)^{\alpha-\sigma^2}\tilde{k}_t^{\alpha^2-\alpha^3}\tilde{\lambda}_t^{\alpha-\alpha^2}\tilde{\lambda}_{t+1}^{1-\alpha}\right),$$

since $k_{t+1} = \alpha\beta k_t^\alpha \lambda_t$.

$$= \left(\frac{\beta E(\lambda^{-1})}{(\alpha\beta)^\alpha}\right)^2 (\alpha\beta)^{\alpha-\sigma^2}\left(E\left(\tilde{k}_t^{\alpha-\alpha^3}\tilde{\lambda}_t^{1-\alpha^2}\tilde{\lambda}_{t+1}^{1-\alpha}\right) - \right.$$

$$\left. E\left(\tilde{k}_t^{\alpha-\alpha^2}\tilde{\lambda}_t^{1-\alpha}\right)E\left(\tilde{k}_t^{\alpha^2-\alpha^3}\tilde{\lambda}_t^{\alpha-\alpha^2}\tilde{\lambda}_{t+1}^{1-\alpha}\right)\right).$$

Since $\{\lambda_t\}$ is i.i.d. and the fact that k_t is determined in period $t-1$ independent of λ_t or λ_{t+1}, we may equivalently write

$$cov(p_t^b, p_{t+1}^b) = \left(\frac{\beta E(\lambda^{-1})}{(\alpha\beta)^\alpha}\right)^2 (\alpha\beta)^{\alpha-\sigma^2}\left(E(\tilde{k}_t^{\alpha-\alpha^3})E(\tilde{\lambda}_t^{1-\alpha^2})E(\tilde{\lambda}_{t+1}^{1-\alpha})\right.$$

$$\left. - E(\tilde{k}_t^{\alpha-\alpha^2})E(\tilde{\lambda}_t^{1-\alpha})E(\tilde{k}_t^{\alpha^2-\alpha^3})E(\tilde{\lambda}_t^{\alpha-\alpha^2})E(\tilde{\lambda}_{t+1}^{1-\alpha})\right).$$

Let. $f_1(k) = k_t^{\alpha-\alpha^2}$ and $f_2(k) = k_t^{\alpha^2-\alpha^3}$. Both $f_1(k)$ and $f_2(k)$ are increasing functions of k, and $f_1(k)f_2(k) = k_t^{\alpha-\alpha^3}$. By the Harris (1960) inequality,

$$E(\tilde{k}_t^{\alpha-\alpha^3}) \geq E(\tilde{k}_t^{\alpha-\alpha^2})E(\tilde{k}_t^{\alpha^2-\alpha^3}).$$ [18]

Similarly,

$$E(\tilde{\lambda}_t^{1-\alpha^2}) \geq E(\tilde{\lambda}_t^{1-\alpha})E(\tilde{\lambda}_t^{\alpha-\alpha^2}).$$

Thus, $cov(p_t^b, p_{t+1}^b) = \left(\frac{\beta E(\lambda^{-1})}{(\alpha\beta)^\alpha}\right)^2 (\alpha\beta)^{\alpha-\sigma^2} E(\tilde{\lambda}_t^{1-\alpha})$

$$\times\left\{E(\tilde{k}_t^{\alpha-\alpha^3})E(\tilde{\lambda}_t^{1-\alpha^2}) - E(\tilde{k}_t^{\alpha-\alpha^2})E(\tilde{\lambda}_t^{1-\alpha})E(\tilde{k}_t^{\alpha^2-\alpha^3})E(\tilde{\lambda}_t^{\alpha-\alpha^2})\right\} \geq 0 \blacksquare$$

[18] The essence of the Harris (1960) inequality is as follows: for any probability measure μ on R, and increasing functions $f(x)$ and $g(x)$, $\int_R f(x)g(x)d\mu(x) \geq \int_R f(x)d\mu(x) \int_R g(x)d\mu(x)$.

The inequality is strict if $\{\tilde{\lambda}_t\}$ is log-normally distributed.

Equity and risk free debt prices are thus unambiguously mean averting under Property I for any i.i.d. shock process. This result confirms that the notions of mean reversion (under I) and stationarity (the existence of a long run ergodic probability distribution on capital stock – the equity price – to which the economy converges) are not equivalent, and that the distinction arises in the simplest equilibrium macroeconomic models. By Proposition 2.1, equity and risk free debt prices fail to mean revert under Property II. Furthermore, these observations are generic in the sense expressed in the following result:

Proposition 3.2: Consider any equilibrium model of the general form (1) for which the equilibrium investment function $i(k, \lambda)$ is continuous and increasing in both its arguments. Suppose also that the period t price of equity and the period t+1 level of the capital stock coincide (no costs of adjustment). Then $\{p_t^e\}$ will be mean averting.

If $p_t^b = h(k, \lambda)$, where $h(\ )$ is continuous and increasing in both its arguments, then $\{p_t^b\}$ is mean averting as well.

Proof: Direct application of the FKG inequality (Fortunin et al. (1971) or Harris (1960)). See the Technical Appendix. ■

Many of the macroeconomic models to be considered in this paper satisfy the conditions of the above proposition. In light of Proposition 3.2, if theory has much to say about economic reality, it is not entirely surprising that empirical studies have found, at best, weak evidence of mean reversion in equity prices (see the excellent discussion and literature review in Spierdijk and Bikker (2012)).

We next make statements regarding mean reversion (Property I) in the equity return series for this model.

Proposition 3.3: For Model (1), specialized by (11) and (12)

a. $corr(\tilde{r}_t^e, \tilde{r}_{t+1}^e) \leq 0$; i.e., equity returns are mean reverting by Property I.
b. $corr(\tilde{r}_t^b, \tilde{r}_{t+1}^b) > 0$; i.e., bond returns are mean averting by Property I.

Proof:

$$cov(\tilde{r}_t^e, \tilde{r}_{t+1}^e) = cov\left(\alpha \tilde{k}_t^{\alpha-1}\tilde{\lambda}_t, \alpha\left[\alpha\beta\tilde{k}_t^{\alpha}\tilde{\lambda}_t\right]^{\alpha-1}\tilde{\lambda}_{t+1}\right)$$
$$= \alpha^2(\alpha\beta)^{\alpha-1}E(\tilde{\lambda}_{t+1})\{E(\tilde{k}_t^{\alpha^2-1}\tilde{\lambda}_t^{\alpha}) - E(\tilde{k}_t^{\alpha-1}\tilde{\lambda}_t)E(\tilde{k}_t^{\alpha^2-1}\tilde{\lambda}_t^{\alpha-1})\}$$
$$= \alpha^2(\alpha\beta)^{\alpha-1}E(\tilde{\lambda}_{t+1})\{E(\tilde{k}_t^{\alpha^2-1})E(\tilde{\lambda}_t^{\alpha}) - E(\tilde{k}_t^{\alpha-1})E(\tilde{k}_t^{\alpha^2-\alpha})E(\tilde{\lambda}_t)E(\tilde{\lambda}_t^{\alpha-1})\}. \quad (15)$$

We wish first to explore the following constituents of expression (15):

$$E\left(\tilde{k}_t^{\alpha^2-1}\right) \text{ vs. } E\left(\tilde{k}_t^{\alpha-1}\right)E\left(\tilde{k}_t^{\alpha^2-\alpha}\right).$$

These expressions are of the general form

$$E\left(\tilde{k}_t^{\gamma_0+\gamma_1}\right) \text{and } E\left(\tilde{k}_t^{\gamma_0}\right)E\left(\tilde{k}_t^{\gamma_1}\right)$$

where $\gamma_0 < 0, \gamma_1 < 0$. Define $x = \tilde{k}_t^{\gamma_0}$, and $g(x) = x^{(\gamma_1/\gamma_0)}$.

Since $(\gamma_1/\gamma_0) > 0$, $g(x)$ is an increasing function of x, and $g(x) = \tilde{k}_t^{\gamma_1}$

Thus, $E\left(\tilde{k}_t^{\gamma_0+\gamma_1}\right) = E\left(\tilde{x}g(\tilde{x})\right) > E(\tilde{x})E\left(g(\tilde{x})\right) = E\left(\tilde{k}_t^{\gamma_0}\right)E\left(\tilde{k}_t^{\gamma_1}\right)$by the FKG inequality.

Accordingly,

$$E\left(\tilde{k}_t^{\alpha^2-1}\right) \geq E\left(\tilde{k}_t^{\alpha-1}\right)E\left(\tilde{k}_t^{\alpha^2-\alpha}\right).$$

We may thus conclude that expression (14)

$$< \alpha^2(\alpha\beta)^{\alpha-1}E\left(\tilde{\lambda}_{t+1}\right)\left\{E\left(\tilde{\lambda}_t^{\alpha}\right) - E\left(\tilde{\lambda}_t\right)E\left(\tilde{\lambda}_t^{\alpha-1}\right)\right\}$$

$$\leq \alpha^2(\alpha\beta)^{\alpha-1}E\left(\tilde{\lambda}_{t+1}\right)\left\{\left(E\left(\tilde{\lambda}_t\right)\right)^{\alpha} - E\left(\tilde{\lambda}_t\right)E\left(\tilde{\lambda}_t^{\alpha-1}\right)\right\}$$

by Jensen's inequality since $g(\lambda) = \lambda^{\alpha}, 0 < \alpha < 1$ is a concave function of λ.

$$\leq \alpha^2(\alpha\beta)^{\alpha-1}E\left(\tilde{\lambda}_{t+1}\right)\left\{\left(E\left(\tilde{\lambda}_t^{\alpha}\right)\right)^{\alpha} - E\left(\tilde{\lambda}_t\right)\left(E\left(\tilde{\lambda}_t\right)\right)^{\alpha-1}\right\},$$

again by Jensen's inequality, since $g(\lambda) = \lambda^{\alpha-1}$ is a convex function of λ.

$$= \alpha^2(\alpha\beta)^{\alpha-1}E\left(\tilde{\lambda}_{t+1}\right)\left\{\left(E\left(\tilde{\lambda}_t\right)\right)^{\alpha} - E\left(\tilde{\lambda}_t\right)\left(E\left(\tilde{\lambda}_t\right)\right)^{\alpha-1}\right\} = 0.$$ [19]

See the Technical Appendix for the analogous proof that $cov\left(\tilde{r}_t^b, \tilde{r}_{t+1}^b\right) \geq 0$ ■

Clearly it is the concavity of the production function $(\alpha - 1) < 0$ that gives mean reversion (as characterized by Property I) first and foremost in equity returns, a fact first observed in Basu and Vinod (1994). Risk-free returns are mean-averting, however. Taken together, Propositions 3.1 and 3.3. remind us that mean reversion in security returns need not arise solely from mean reversion in prices.

It remains to see if equity prices or returns in the Benchmark model are mean reverting by Property III. Proposition 2.2 (see Footnote 8) informs us that the $var(k_t)$ and the $var(r_t^e)$ will be the critical factors in the analysis. By (12), the expression for the variance of the equity price becomes:

$$var(\tilde{p}_{t-1}^e) = var\left(\tilde{k}_t\right) = \left[(\alpha\beta)^{1+\alpha+\alpha^2+\cdots+\alpha^{t-1}}\right]^2\left[k_0^{\alpha^t}\right]^2 var\left(\prod_{s=0}^{t-1}\tilde{\lambda}_s^{\alpha^{t-1-s}}\right) \tag{16}$$

[19] We thank Awi Federgruen for bringing the FKG inequality to our attention.

where $var\left(\prod_{s=0}^{t-1}\tilde{\lambda}_s^{\alpha^{t-1-s}}\right) = var\left[\prod_{s=0}^{t-1}\tilde{\lambda}_s^{\alpha^s}\right]$ by the i.i.d. assumption on the $\{\tilde{\lambda}_t\}$. By the same analysis,

$$\operatorname{var} r_t^e = \operatorname{var} \alpha k_t^{\alpha-1}\lambda_t, \text{ where}$$

$$r_t^e = \alpha\left[(\alpha\beta)^{\frac{1}{1-\alpha}(1-\alpha^{t-1})}k_0^{\alpha^t}\prod_{s=0}^{t-1}\tilde{\lambda}_s^{\alpha^s}\right]^{\alpha-1}\tilde{\lambda}_t - 1$$

$$= \alpha\tilde{\lambda}_t\left[(\alpha\beta)^{-(1-\alpha^{t-1})}k_0^{\alpha^t(\alpha-1)}\prod_{s=0}^{t-1}\tilde{\lambda}_s^{\alpha^s(\alpha-1)}\right] - 1.$$

Accordingly,

$$var(\tilde{r}_t^e) = \alpha^2 E\left(\tilde{\lambda}_t\right)^2\left[(\alpha\beta)^{-(1-\alpha^{t-1})}k_0^{\alpha^t(\alpha-1)}\right]^2\prod_{s=0}^{t-1}E\left(\tilde{\lambda}_s^{\alpha^s(\alpha-1)}\right)^2$$

$$-\alpha^2\left(E\left(\tilde{\lambda}_t\right)\right)^2\left[(\alpha\beta)^{-(1-\alpha^{t-1})}k_0^{\alpha^t(\alpha-1)}\right]^2\prod_{s=0}^{t-1}\left[E\left(\tilde{\lambda}_s^{\alpha^s(\alpha-1)}\right)\right]^2 \text{[20]}$$

$$= \alpha^2\left(E\left(\tilde{\lambda}_t\right)\right)^2\left[(\alpha\beta)^{-2(1-\alpha^{t-1})}k_0^{2\alpha^t(\alpha-1)}\right]^2$$

$$\left\{\prod_{s=0}^{t-1}E\left(\tilde{\lambda}_s^{\alpha^s(\alpha-1)}\right)^2 - \prod_{s=0}^{t-1}\left[E\left(\tilde{\lambda}_s^{\alpha^s(\alpha-1)}\right)\right]^2\right\}. \qquad (17)$$

Let us slightly specialize the productivity to be of the business-cycle-literature-inspired form $\{e^{\tilde{\lambda}_t}\}$, where $\tilde{\lambda}_{t+1} = \rho\tilde{\lambda}_t + \tilde{\varepsilon}_{t+1},\ \{\tilde{\varepsilon}_t\}$ i.i.d. $N(0,\sigma_\varepsilon^2)$. For Property III, we have the following proposition.

Proposition 3.4: Consider Model 1 specialized as per (11) and (12) cum an i.i.d. shock process of the type noted above. Then,

(i) The equity price series and dividend series are mean averting by Property III;

(ii) The return on equity and equity premium series are mean reverting by Property III.

Proof: See the Technical Appendix.

These results are entirely consistent with those obtained for our earlier analysis of Properties I (and, by implication, Property II).

A check of the proof of Proposition 3.4 reveals that concavity in production $(\alpha < 1)$ is, once again, the overriding guarantor of mean reversion, though in a somewhat indirect way arising as it does not through the productivity disturbance but

[20] Here we use the following property of independent random variables: $var(\tilde{x}\tilde{y}) = E(\tilde{x}^2)E(\tilde{y}^2) - (E\tilde{x})^2(E\tilde{y})^2$

through the repeated influence of the capital share and discount factor parameters on capital accumulation.

Propositions 3.1 – 3.3 give cause for reflection. If the notion of mean reversion is intended to capture the property that above average values of a stochastic process must regularly be followed by below average values, then all the series considered thus far, $\{\tilde{p}_t^e\}$, $\{\tilde{d}_t\}$, $\{\tilde{r}_t^e\}, \{\tilde{p}_t^b\}$, and $\{\tilde{r}_t^b\}$ qualify: each follows a stationary stochastic process that converges to a unique, irreducible ergodic set. Accordingly, the commonplace definitions of mean reversion in the literature (Properties I – II) seem to discriminate artificially: both
$\{\tilde{r}_t^e\}$ and $\{\tilde{p}_t^e\} = \{\tilde{k}_t\}$ revert to their respective means in the intuitive sense of those words, yet behave entirely inconsistently as regards Properties I and II.

3.2. An Alternative Measurement

Informally we think of a mean reverting series $\{\tilde{x}_t\}$ as one whose value "crosses its mean 'fairly frequently'." As we have shown, the requirement that $corr(\tilde{x}_t, \tilde{x}_{t+1}) < 0$ is, however, too restrictive to be exlusively identified with this concept of mean reversion. A more informative (and intuitive) measurement is needed. Implicit in this comment is the desire also to have an intuitive measure by which one series can be said to be more strongly mean reverting than another.

Let us recall to this discussion an old notion of a stochastic process's average time to crossing (ACT): the average number of periods for which a stochastic process uniformly exceeds or uniformly falls short of its mean value. Under this concept an economic time series is mean reverting if and only if its ACT is finite. Within the family of models under consideration in this paper, it is also natural to say that a stochastic series $\{\tilde{x}_t\}$ is more highly mean reverting than a stochastic series $\{\tilde{y}_t\}$ if and only if the $ACT_{\{\tilde{x}_t\}} < ACT_{\{\tilde{y}_t\}}$. Below we list correlations and ACTs for the Benchmark model and calibration previously discussed.

Table 3.4

Benchmark Model: Correlations and ACTs

$\alpha = 0.36, \beta = 0.99, \Omega = 1, \{\tilde{\lambda}_t\}$ $i.i.d., \sigma_\varepsilon^2 = 0.00712$

$corr(p_t^e, p_{t+1}^e)$	$corr(p_t^b, p_{t+1}^b)$	$corr(r_t^e, r_{t+1}^e)$	$corr(r_t^b, r_{t+1}^b)$	$corr(r_t^p, r_{t+1}^p)$
.3868	.3870	-.3020	.3879	.0300
$ACT(p_t^e)$	$ACT(p_t^b)$	$ACT(r_t^e)$	$ACT(r_t^b)$	$ACT(r_t^p)$
2.696	2.697	1.658	2.691	2.02

Note that the ordering of correlations (smallest to largest) and ACTs is the same: subject to rounding/numerical approximations, a more positive autocorrelation is associated with a larger ACT, which implies less frequent "crossings." Proposition 3.5 formalizes this observation.

Proposition 3.5: Let $\{\tilde{x}_t\}$ and $\{\tilde{y}_t\}$ be two stationary stochastic processes representing equilibrium state/decision variables arising from a model of the type (6). Then $corr(\tilde{x}_t, \tilde{x}_{t+1}) > corr(\tilde{y}_t, \tilde{y}_{t+1})$ if and only if $ACT_{\{\tilde{x}_t\}} < ACT_{\{\tilde{y}_t\}}$. Furthermore, for each value of $corr(x_v, x_{t+1})$ there is a unique $ACT_{\{\tilde{x}_t\}}$.

Proof: See the Technical Appendix.

Some results in Table 3.4 are unique to the Benchmark parameterization (in particular to the $\rho = 0, \Omega = 1$ assumptions). In particular $corr(p_t^e, p_{t+1}^e) = corr(p_t^b, p_{t+1}^b)$ and thus $ACT_{\{p_t^e\}} = ACT_{\{p_t^b\}}$. These identities follow from the fact that $p_t^b = \frac{E(\lambda_t^{-1})}{\alpha}(p_t^e)^{1-\alpha}$. Accordingly, $\{p_t^b\}$ exceeds its mean when p_t^e does and vice versa. With identical ACTs, their autocorrelations must be identical. Furthermore, since , $r_t^b = \frac{1}{p_t^b} - 1, r_t^b$ will exceed its mean if and only if p_t^b falls short of its mean, and vice versa, leading to identical ACTs for $\{p_t^b\}$ and $\{r_t^b\}$. These relationships do not, however, necessarily apply to more general versions of the Benchmark ($\rho = 0$ is necessary). Note that the premium is very slightly positively autocorrelated, which is a fair approximation to actual data.

Let us in this context reinforce our earlier remarks concerning the commonplace characterizations of mean reversion. The results of Table 3.4 clearly suggest that to identify a mean reverting series exclusively with negative autocorrelation is a false identification: All the series in Table 3.4 mean revert (they have finite ACT measurements) yet only on $\{r_t^e\}$ it is negatively autocorrelated. Negative autocorrelation means "frequent crossings (of the mean)," nothing more.

The Benchmark model falls short, however, of a full-fledged business cycle model on many dimensions. In particular, none of the aggregate series is sufficiently persistent vis-à-vis the data. In the next section we remedy this particular shortcoming and explore the consequences of this and other model generalizations for mean reversion.

4. Model Generalizations

In this section we explore the implications for mean reversion in security prices and returns of adding a variety of model features: (1) persistence in the productivity shock, (2) greater concavity in the representative agent's period utility function, (3) the addition of a labor-leisure choice and (4) the addition of a cost-of-adjustment function on the representative firm's various return series.

4.1 Adding Persistence in the Productivity Disturbance

In conformity with the macroeconomics DSGE literature in this section we specialize the production technology to be of the form $y_t = \tilde{k}_t^{\alpha} e^{\tilde{\lambda}_t}$ where and $\tilde{\lambda}_{t+1} = \rho\tilde{\lambda}_t + \tilde{\varepsilon}_{t+1}$, $\tilde{\varepsilon}_t$ is i.i.d., $\tilde{\varepsilon}_t \sim N(0, \sigma_\varepsilon^2)$, and $\rho > 0$. Despite the added persistence in the productivity disturbance, the decision rules take the same form as (13) – (14).[21]

The addition of persistence also does not alter the expressions for p_t^e and r_t^e; p_t^b and r_t^b are slightly modified, however:

$$p_t^b = \frac{\beta}{(\alpha\beta)^\alpha} e^{\sigma_\varepsilon^2/2} k_t^{\alpha-\alpha^2} e^{(1-\alpha-\rho)\lambda_t}, \text{ and}$$

$$r_t^b = \frac{(\alpha\beta)^\alpha}{\beta} e^{-\sigma_\varepsilon^2/2} k_t^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\lambda_{t-1}} - 1.$$

---

[21] The necessary and sufficient conditions for the optimal investment function is: $u_1(c_t) = \beta \int u_1(\tilde{c}_{t+1}) \alpha k_{t+1}^{\alpha-1} e^{\tilde{\lambda}_{t+1}} dF(\lambda_{t+1}, \lambda_t)$. For the indicated functional forms and decision rules, this equation becomes:

$$\frac{1}{(1-\alpha\beta)k_t^\alpha e^{\lambda_t}} = \beta \int \frac{\alpha\left(\alpha\beta k_t e^{\lambda_t}\right)^{\alpha-1} e^{\rho\lambda_t + \tilde{\varepsilon}_{t+1}}}{(1-\alpha\beta)(\alpha\beta k_t e^{\lambda_t})^\alpha e^{\rho\lambda_t+\tilde{\varepsilon}_{t+1}}} dF(\tilde{\varepsilon}_{t+1})$$

which reduces to $1 = \int dF(\tilde{\varepsilon}_{t+1}) = 1$.

With regard to prices, our results mirror their earlier counterparts.

Proposition 4.1: For the Benchmark model where $\rho > 0$, the equity price series, $\{\tilde{p}_t^e\}$, the dividend series $\{\tilde{d}_t\}$, and the risk free asset price series $\{\tilde{p}_t^b\}$ are all mean averting by Property I.

Proof: We offer the proof only for $\{\tilde{p}_t^e\}$; $\{\tilde{d}_t\}$ and $\{\tilde{p}_t^b\}$ are analyzed similarly.

$$\begin{aligned} cov(\tilde{p}_t^e, \tilde{p}_{t+1}^e) &= cov(\tilde{k}_{t+1}^e, \tilde{k}_{t+2}^e) = cov\left(\alpha\beta\tilde{k}_t^\alpha e^{\tilde{\lambda}_t}, \alpha\beta\left[\alpha\beta\tilde{k}_t^\alpha e^{\tilde{\lambda}_t}\right]^\alpha e^{\tilde{\lambda}_{t+1}}\right) \\ &= cov\left(\alpha\beta\tilde{k}_t^\alpha e^{\tilde{\lambda}_t}, (\alpha\beta)^{1+\alpha}\tilde{k}_t^{\alpha^2} e^{\alpha\tilde{\lambda}_t} e^{\rho\tilde{\lambda}_t+\tilde{\varepsilon}_{t+1}}\right) \\ &= (\alpha\beta)^{2+\alpha} cov\left(\tilde{k}_t^\alpha e^{\tilde{\lambda}_t}, \tilde{k}_t^{\alpha^2} e^{(\alpha+\rho)\tilde{\lambda}_t} e^{\tilde{\varepsilon}_{t+1}}\right) \\ &= (\alpha\beta)^{2+\alpha} E\left(e^{\tilde{\varepsilon}_{t+1}}\right)\left\{E\left(\tilde{k}_t^{\alpha+\alpha^2} e^{1+(\alpha+\rho)\tilde{\lambda}_t}\right) - \right. \\ &\left. E\left(\tilde{k}_t^\alpha e^{\tilde{\lambda}_t}\right) E\left(\tilde{k}_t^{\alpha^2} e^{(\alpha+\rho)\tilde{\lambda}_t}\right)\right\}. \end{aligned}$$

Let $g^1(k_t, \lambda_t) = \tilde{k}_t^\alpha e^{\tilde{\lambda}_t}$ $\qquad g_1^1 > 0; g_2^1 > 0$

$g^2(k_t, \lambda_t) = \tilde{k}_t^{\alpha^2} e^{(\alpha+\rho)\tilde{\lambda}_t}$ $\qquad g_1^2 > 0; g_2^2 > 0$

➔ by Harris (1960)

$$E\left(g^1(k,\lambda) g^2(k,\lambda)\right) \geq E\left(g^1(k,\lambda)\right) E\left(g^2(k,\lambda)\right)$$

Thus, $cov(\tilde{p}_t^e, \tilde{p}_{t+1}^e) = cov(\tilde{k}_{t+1}^e, \tilde{k}_{t+2}^e) \geq 0$. ■

In passing to returns, we first rely on numerical simulations of (11) – (14) to give us some measure of the relevant magnitudes.

Table 4.1
Model 2: Autocorrelations, ACTs (i)
$u(c) = \log(c), \alpha = 0.36, \beta = 0.99, \Omega = 1$
$\tilde{\lambda}_{t+1} = \rho\tilde{\lambda}_t + \tilde{\varepsilon}_{t+1}, \sigma_\varepsilon^2 = 0.00712$ (ii)

Panel A: Autocorrelations: Various ρ

	$\rho = 0$	$\rho = 0.2$	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$	$\rho = 0.95$
$corr(p_t^e, p_{t+1}^e)$	.3868	.5436	.6792	.7929	.9031	.9759
$corr(p_t^b, p_{t+1}^b)$	.3870	.6315	.8039	.8365	.6746	.4590
$corr(r_t^e, r_{t+1}^e)$	-.3020	-.1630	-.0236	.1155	.2532	.3546
$corr(r_t^b, r_{t+1}^b)$	.3879	.6319	.8039	.8366	.6750	.4589
$corr(r_t^p, r_{t+1}^p)$	.0300	.0299	.0298	.0297	.0295	.0294

Panel B: ACTs

	$\rho = 0$	$\rho = 0.2$	$\rho = 0.4$	$\rho = 0.6$	$\rho = 0.8$	$\rho = 0.95$
$ACT(p_t^e)$	2.70	3.14	3.83	4.91	7.20	13.89
$ACT(p_t^b)$	2.70	3.52	4.88	5.39	3.71	2.84
$ACT(r_t^e)$	1.66	1.80	1.97	2.17	2.41	2.60
$ACT(r_t^b)$	2.69	3.512	490	5.40	3.72	2.84
$ACT(r_t^p)$	2.02	2.02	2.02	2.02	2.02	2.02

(i) Statistics based on time series of length 10,000
(ii) The numbers reported in this table are unaffected by the magnitude of σ_ε. They are also unaffected by the choice of $\beta > 0$, provided $\beta < 1$.

While the results of Table 4.1–Panel A for equity returns certainly respect the implications of Propositions 3.3 in the $\rho = 0$ case, the conclusion is not robust: adding sufficient persistence to the random productivity disturbance causes the equity return series to become mean averting.

The results of Table 4.1 for $\{r_t^b\}$ are partially rationalized in Proposition 4.2.

<u>Proposition 4.2</u>: Consider the model defined by (11), (12) with shock process and production technology specialized to $y_t = \tilde{k}_t^\alpha e^{\tilde{\lambda}_t}$ where and $\tilde{\lambda}_{t+1} = \rho\tilde{\lambda}_t + \tilde{\varepsilon}_{t+1}$, $\{\tilde{\varepsilon}_t\}$ is i.i.d., $N(0, \sigma_\varepsilon^2)$. A sufficient condition for $\{\tilde{r}_t^e\}$ and $\{\tilde{r}_t^b\}$ to be mean averting by Property I is that $\alpha + \rho > 1$.

Proof:

$$cov(r_t^b, r_{t+1}^b) = cov(r_{t+1}^b, r_{t+2}^b)$$

$$= \left[\frac{(\alpha\beta)^\alpha}{\beta} e^{-\sigma_\varepsilon^2/2}\right]^2 cov\left(k_t^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\lambda_t}, k_{t+1}^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\lambda_{t+1}}\right)$$

Let $L = \left[\frac{(\alpha\beta)^\alpha}{\beta} e^{-\sigma_{\tilde{\varepsilon}}^2/2}\right]^2$

But $k_{t+1} = \alpha\beta k_t^\alpha e^{\lambda_t}$ and $\tilde{\lambda}_{t+1} = \rho\tilde{\lambda}_t + \tilde{\varepsilon}_{t+1}$; thus:

$$= L(\alpha\beta)^{\alpha^2-\alpha} cov\left(k_t^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\lambda_t}, k_t^{\alpha^3-\alpha^2} e^{\lambda_t(\alpha^2-\alpha)} e^{(\alpha+\rho-1)\lambda_t(\alpha+\rho-1)\tilde{\varepsilon}_{t+1}}\right)$$

$$= L(\alpha\beta)^{\alpha^2-\alpha} cov\left(k_t^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\lambda_t}, k_t^{\alpha^3-\alpha^2} e^{[(\alpha^2-\alpha)+(\alpha+\rho-1)\rho]\lambda_t} e^{(\alpha+\rho-1)\tilde{\varepsilon}_{t+1}}\right)$$

$$= L(\alpha\beta)^{\alpha^2-\alpha} e^{(\alpha+\rho-1)^2\sigma_{\tilde{\varepsilon}}^2/2}\left\{E\left(k_t^{\alpha^3-\alpha} e^{[(\alpha^2-\alpha)+2(\alpha+\rho-1)]\lambda_t}\right) - E\left(k_t^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\lambda_t}\right) E\left(k_t^{\alpha^3-\alpha^2} e^{[(\alpha^2-\alpha)+(\alpha+\rho-1)\rho]\lambda_t}\right)\right\}$$

Let $g^1(k;\lambda) = k_t^{\alpha^2-\alpha} e^{(\alpha+\rho-1)\lambda_t}$

$g^2(k;\lambda) = k_t^{\alpha^3-\alpha^2} e^{[(\alpha^2-\alpha)+(\alpha+\rho-1)\rho]\lambda_t}$

$g_1^1(k;\lambda) < 0 \quad g_2^1(k;\lambda) < 0$ if $\alpha + \rho < 1$

$g_1^2(k;\lambda) < 0 \quad g_2^2(k;\lambda) < 0$ if $\alpha + \rho < 1$

same is true for $g^1(k;\lambda)g^2(k;\lambda)$

So $-g_1(k;\lambda)$ is increasing

$-g_2(k;\lambda)$ is increasing.

$$\int -g_1(k;\lambda)\left(-g_2(k;\lambda)\right) dF(k;\lambda) \geq \int -g_1(k;\lambda) dF(k;\lambda) \int -g_2(k;\lambda) dF(k;\lambda)$$

$$\int g_1(k;\lambda) g_2(k;\lambda) dF(k;\lambda) \geq \int g_1(k;\lambda) dF(k;\lambda) \int g_2(k;\lambda) dF(k;\lambda)$$

by Harris (1960).

Thus, $cov\left(r_t^b, r_{t+1}^b\right) \geq 0$. ■

Table 4.1 and Proposition 4.1 introduce persistence in the productivity disturbances into our Benchmark Model in a simple way typical of the DSGE literature. The conclusion is that if these productivity disturbances are sufficiently persistent, equity returns will be mean averting while risk free returns are always so. As a consequence, if a model of this sort even is to come close to matching the observed persistence in output, equity returns will surely be mean averting at least by Definition I. (As per Proposition 2.1, returns will be similarly mean averting by Property II for sufficiently high persistence.)

None of these results is surprising in the least: the process on the disturbance component, $\{e^{\tilde{\lambda}_t}\}$, is itself highly mean averting as the following proposition makes clear. Cogley and Nason (1995) emphasize the close relationship of the productivity process to the derived properties of DSGE models' state variables.

Proposition 4.3: Consider a stochastic process of the form

$\tilde{x}_t = \rho x_{t-1} + \tilde{\varepsilon}_t$, where $\{\tilde{\varepsilon}_t\}$ is i.i.d. with mean zero and variance σ^2. Define a new stochastic process by

$\tilde{\lambda}_t = e^{\tilde{x}_t}$.

Then,

$$cov(\tilde{\lambda}_t, \tilde{\lambda}_{t+1}) = e^{\left(\frac{\sigma}{2}\left(1+(\rho^2+1)\right)\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)}\left(e^{\rho\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)} - 1\right)$$

and, if $-1 < \rho < 1$,

$$cov(\tilde{\lambda}_t, \tilde{\lambda}_{t+1})\begin{Bmatrix} > 0 \; if \; \rho > 0 \\ < 0 \; if \; \rho < 0 \end{Bmatrix}.$$

Proof: See the Appendix.

By Proposition 4.3, the shock process to our production technology is mean reverting by Property I only if $\rho < 0$, an attribution antithetical to its counterpart, the Solow residual, and this feature drags mean aversion into equity returns as well. In this light, it would be useful to understand what additional model features allow high persistence in aggregate series (as the data reveals) to be compatible with mean reversion in equity returns and the equity premium.

As shock persistence increases ($\rho > 0$), however, our earlier results for Property III (Proposition 3.4) are weakened for prices: Table 4.2 summarizes the results of extensive numerical simulations that compute $corr(\tilde{x}_s - \tilde{x}_r, \tilde{x}_u - \tilde{x}_t)$ for a wide class of $\{r, s, t, u\}$ where $r < s < t < u$, and $\{\tilde{x}_t\}$ chosen from $\{\tilde{r}_t^e\}, \{\tilde{r}_t^p\}, \{\tilde{p}_t^e\}, \{\tilde{d}_t\}$.

Table 4.2

Correlations: Various Series
Simulation Results for s = r+i, t = s+j, u = t+k
$i, j, k \in \{1,2,3\}$
$\alpha = 0.36, \beta = 0.96, \sigma_\varepsilon^2 = 0.00712, \rho = 0.95$

Series Correlation	Range of Values across all i, j, k
(i) $corr(p_s^e - p_r^e, p_u^e - p_t^e)$ ambig.	(–.08, +.07)
(ii) $corr(r_s^e - r_r^e, r_u^e - r_t^e) < 0$	(–.18, .00)
(iii) $corr\left(r_s^p - r_r^p, r_u^p - r_t^p\right) < 0$	(–.15, .00)

While these results are consistent (with Proposition 3.4) with those concerning $\{\tilde{r}_t^e\}$ and $\{\tilde{r}_t^p\}$ for Properties I and II, they are distinctly different for the $\{\tilde{p}_t^e\}$ and $\{\tilde{d}_t\}$ series, a fact that accounts for our earlier comment that Property III represents a different measurement from either Property I or II. The switch occurs at around $\rho = 0.6$ for $\alpha = 0.36$ and $\beta = 0.96$ and $\sigma_\varepsilon^2 = 0.00712$, our customary business cycle parameters.

We close Section 4 with a short summary of what we have learned: First, persistence in the productivity disturbance generically overturns the results relative to the case of independence. Consistent with all three of our definitions, equity returns appear necessarily to be mean reverting only in the presence of low persistence productivity disurbances. While it is already well known that mean reversion in returns is compatible with mean aversion in prices (Spierdijk and Bikker (2012)), it is surprising to find this compatibility in such a simple equilibrium context. Proposition 4.3 further suggests that this particular phenomenon is likely to be pervasive across many DSGE formulations, implying that the search for mean reversion in equity returns and the premium is unlikely to be fruitful – if the present family of models has anything to say about actual economies. To say it differently, we find it unsurprising from a theoretical perspective, that evidence for mean revisersion in equity returns and the premium is weak (see Table 4.3).

Table 4.3

First Order Autocorrelations: Annual Returns on the S&P$_{500}$, Various Historical Periods

Historical Period	$\text{corr}\left(r_t^{S\&P_{500}},\ r_{t+1}^{S\&P_{500}}\right)$
1900 – 2014	−.011
1900 – 2012	−.011
1926 – 1996	.289
1952 – 2006	.276
1952 – 2014	−.088

4.2 Incomplete Depreciation

In this section we modify the model of Section 4.1 in only one way to admit partial depreciation. As a result the equation of motion on capital stock becomes:

$$k_{t+1} = (1 - \Omega)k_t + i_t.$$

With this change, precise formulae for the risk free bond price and its rate of return are, however, lost. As in Section 4.1, we therefore rely exclusively on numerical simulation. Table 4.4 summarizes the results:

Table 4.4
Autocorrelations, ACTs
Baseline Case
Various Ω

	$\Omega = 1$			$\Omega = 0.4$			$\Omega = 0.025$		
	$\rho = 0$	$\rho = .4$	$\rho = .95$	$\rho = 0$	$\rho = .4$	$\rho = .95$	$\rho = 0$	$\rho = .4$	$\rho = .95$
$corr(p_t^e, p_{t+1}^e)$	.3868	.6792	.9759	.6681	.8401	.9889	.9653	.9846	.999
$corr(p_t^b, p_{t+1}^b)$	.3870	.8039	.4590	.6685	.869	.6734	.9653	.5969	.9287
$corr(r_t^e, r_{t+1}^e)$	-.3020	-.0236	.3546	-.1341	.202	.6338	.0209	.4137	.9253
$corr(r_t^b, r_{t+1}^b)$	.3879	.8039	.4569	.6688	.8690	.6730	.9653	.5970	.9287
$corr(r_t^p, r_{t+1}^p)$	.0300	.0298	.0294	0299	.0297	.0291	.0296	.0291	.0291
$corr(c_t, c_{t+1})$	.3868	.6792	.9759	.6686	.8199	.9843	.9653	.9787	.9956
$corr(k_t, k_{t+1})$	.3868	.6792	.9759	.6681	.8401	.9889	.9653	.9787	.9956

Panel B: ACTs

$ACT(p_t^e)$	2.696	3.827	13.89	3.77	5.49	19.92	11.31	16.89	74.63
$ACT(p_t^b)$	2.697	4.876	2.84	3.78	6.11	3.81	11.19	3.35	8.51
$ACT(r_t^e)$	1.658	1.968	2.61	1.83	2.32	3.58	2.01	2.737	8.33
$ACT(r_t^b)$	2.691	4.898	2.84	3.77	6.07	3.81	11.22	3.35	8.52
$ACT(r_t^p)$	2.02	2.02	2.02	2.02	2.02	2.02	2.02	2.02	2.02

The message of Panels A and B of Table 4.4 is unambiguous: lower depreciation rates (smaller Ω) increase autocorrelations for all price and returns series except the premium which is largely unaffected. Compatible results are found in the ACT measurements. When $\Omega = 0.025$ all series become positively autocorrelated, even for $\{\tilde{r}_t^e\}$ when $\rho = 0$, in contrast to the conclusions of Proposition 3.3 (which applies only to the $\Omega = 1$ case).

Why is this observed? The answer can be found in the manner by which changing depreciation rates affect the capital stock series and the representative agent's consumption series: both become much larger and the variation introduced by the productivity disturbance becomes proportionately smaller. As a result the autocorrelation of both series becomes much more positive as indicated at the bottom of Panel A. If capital stock becomes more highly autocorrelated so also must be the series $\{\tilde{p}_t^e\}$ and $\{\tilde{r}_t^e\}$. If consumption becomes more highly autocorrelated, so also will the risk free bond price series and the risk free return. The lack of consequences for the premium is as noted earlier.

4.3 Higher Degrees of Risk Aversion

Here we expand the basic model to include period preference orderings captured by $u(c_t) = \frac{(c_t)^{1-\eta}}{1-\eta}$, various $\eta > 1$ In these situations, the representative agent's CRRA is also his EIS; accordingly, the agent prefers less intertemporal consumption variation for higher values of η.

Table 4.5
Autocorrelations, ACTs
Baseline Case: Higher Risk Aversion

$\eta = 2$

Panel A: Autocorrelations

	$\Omega = 1$			$\Omega = 0.4$			$\Omega = 0.025$		
	$\rho = 0$	$\rho = .4$	$\rho = .95$	$\rho = 0$	$\rho = .4$	$\rho = .95$	$\rho = 0$	$\rho = .4$	$\rho = .95$
$corr(p_t^e, p_{t+1}^e)$	.5024	.749	.982	.751	.883	.992	.976	.989	.999
$corr(p_t^b, p_{t+1}^b)$	.5027	.837	.634	.751	.911	.800	.976	.685	.953
$corr(r_t^e, r_{t+1}^e)$	-.1833	.122	.480	-.015	.345	.741	.044	.443	.948
$corr(r_t^b, r_{t+1}^b)$	.5038	.837	.634	.752	.911	.800	.976	.685	.953
$corr(r_t^p, r_{t+1}^p)$	.0302	.030	.029	.030	.030	.029	.030	.029	.029
$corr(c_t, c_{t+1})$	.5027	.7159	.9739	.7512	.8462	.9809	.9764	.9830	.9941

Panel B: ACTSs

ACT(p_t^e)	2.99	4.35	16.08	4.37	6.43	23.15	14.02	22.52	88.50
ACT(p_t^b)	2.99	5.44	3.54	4.36	7.49	4.78	14.29	3.82	10.15
ACT(r_t^e)	1.77	2.18	2.95	1.97	2.58	4.29	2.05	2.81	9.93
ACT(r_t^b)	2.99	5.45	3.54	4.36	7.51	4.77	14.33	3.81	10.19
ACT(r_t^p)	2.02	2.02	2.02	2.02	2.02	2.02	2.02	2.02	2.02

$\eta = 5$

Panel A: Autocorrelations

$corr(p_t^e, p_{t+1}^e)$	.643	.827	.988	.834	.924	.995	.987	.994	.999
$corr(p_t^b, p_{t+1}^b)$	.643	.879	.845	.835	.945	.927	.986	.793	.978
$corr(r_t^e, r_{t+1}^e)$	.082	.421	.694	.188	.557	.883	.091	.502	.974
$corr(r_t^b, r_{t+1}^b)$	.644	.880	.845	.835	.945	.927	.986	.793	.978
$corr(r_t^p, r_{t+1}^p)$	.030	.030	.029	.029	.030	.029	.029	.029	.029
$corr(c_t, c_{t+1})$	.6433	.7705	.9711	.8345	.8823	.9777	.9864	.9888	.9941

Panel B: ACTSs

ACT(p_t^e)	3.60	5.25	19.38	5.40	8.16	30.49	21.79	31.16	129.88
ACT(p_t^b)	3.61	6.31	5.46	5.40	9.71	7.96	21.79	4.76	15.18
ACT(r_t^e)	2.11	2.74	3.91	2.28	3.15	6.31	2.09	2.99	13.72
ACT(r_t^b)	3.61	6.32	5.45	5.44	9.65	7.95	21.88	4.77	15.13
ACT(r_t^p)	2.02	2.02	2.02	2.02	2.02	2.02	2.02	2.02	2.02

If we compare Table 4.1 (Baseline) with Table 4.5 ($\eta = 2{,}5$), entry by entry we see that higher risk aversion, ceteris paribus, increases autocorrelations across all return and price series except for the premium. To understand the mechanism, it is necessary to recall that for the simple preference orderings studied in the paper, the CRRA and the EIS of the representative agent are identical. Accordingly, greater representative agent risk aversion translates into the desire for a smoother intertemporal consumption stream, and the representative agent goes about effecting the economy's investment policy to bring this about: Comparing Tables 4.1 to 4.4, higher risk aversion is seen to lead to higher consumption autocorrelation. It directly follows that the price of one unit of consumption next period, the risk free bond price, will become more highly autocorrelated as well. Following our earlier remarks, it further follows naturally that risk free returns will also become more highly autocorrelated (see the earlier argument in Section 4.1).

On the equity side, in order to promote a smoother consumption path, the path of capital stock must be more stable intertemporally – more positively autocorrelated, at the expense of greater investment volatility (to which the representative agent is indifferent). Ceteris paribus, the equity return series becomes more highly autocorrelated, as evidenced in the data. In summary, within the CRRA class of preference orderings, greater risk aversion discourages mean reversion as it is commonly measured. The increased ACTs that accompany the increased g , case by case, represent another perspective on this same truth.

Comparing the autocorrelations of, e.g., the equity price series where $\Omega = 0.025$, as shock autocorrelation increases from $\rho = 0.4$ to $\rho = 0.95$ we see that $corr(p_t^e, p_{t+1}^e)$ increases from 0.989 to 0.999 which seems insignificant. The corresponding ACTs are, respectively, 22.52 and 88.50 quarters, however, which is a dramatic and, in our opinion, more informative measurement. It is in this sort of comparison that the ACT measurement, in our judgment, is more revealing than pure autocorrelation statistics.

4.4 Habit Formation

In this section we modify preferences to add an external habit. In the Baseline case, this means modifying the representative agent's period utility function to be of the form $u(c_t - \psi c_{t-1}) = \log(c_t - \psi c_{t-1})$; with higher risk aversion CRRA utilities are modified similarly. It is well known that habit formation causes the agent to behave in a more risk averse fashion. Following the conclusions of Section 4.3, we would expect that its addition would increase autocorrelations and ACTs across the board (all cases of returns and prices except for the premium). This is apparent if Table 4.6 is compared with Table 4.4 entry by entry. The discussion following Table 4.5 applies here as well: more risk averse agents act to stabilize consumption and capital intertemporally with the indicated consequences. For higher degrees of risk aversion, the effects compound.

Table 4.6
Autocorrelations, ACTs

Baseline Case with Habit Formation
$u(c_t - \psi c_{t-1}) = \log(c_t - \psi c_{t-1}), \psi = 0.86$

Panel A: Autocorrelations

	$\Omega = 1$			$\Omega = 0.4$			$\Omega = 0.025$		
	$\rho = 0$	$\rho = .4$	$\rho = .95$	$\rho = 0$	$\rho = .4$	$\rho = .95$	$\rho = 0$	$\rho = .4$	$\rho = .95$
$corr(p_t^e, p_{t+1}^e)$	.6096	.7670	.9496	.8054	.8935	.9813	.9729	.9874	.9989
$corr(p_t^b, p_{t+1}^b)$	.6100	.8009	.8931	.8058	.9145	.9023	.9728	.6628	.9343
$corr(r_t^e, r_{t+1}^e)$	.1389	.4303	.5916	.1759	.5188	.7618	.0373	.4330	.9289
$corr(r_t^b, r_{t+1}^b)$	.6113	.8010	.8930	.8060	.9144	.9025	.9728	.6629	.9343
$corr(r_t^p, r_{t+1}^p)$	.0305	.0303	.0298	.0310	.0299	.0293	.0296	.0293	.0288
$corr(c_t, c_{t+1})$	.8600	.8962	.9845	.9499	.9587	.9918	.9969	.9972	.9990

Panel B: ACTSs

$ACT(p_t^e)$	3.46	4.55	10.03	4.94	6.87	16.34	12.44	18.59	65.79
$ACT(p_t^b)$	3.45	4.91	6.67	4.94	7.63	7.13	12.34	3.681	8.704
$ACT(r_t^e)$	2.17	2.78	3.34	2.26	3.02	4.45	2.03	2.78	8.36
$ACT(r_t^b)$	3.44	4.88	6.76	4.93	7.65	7.12	12.34	3.68	8.78
$ACT(r_t^p)$	2.02	2.02	2.02	2.02	2.02	2.02	2.02	2.02	2.02
$ACT(c_t)$	5.78	6.79	16.72	9.78	10.78	23.15	39.69	42.02	71.44

4.5 Adding a Labor/Leisure Choice

We do this in two ways by specifying the representative agent's period utility specification to be either (1) $u(c_t, 1-n_t) = \log c_t + A\log(1-n_t)$ or (2) $u(c_t, 1-n_t) = \frac{[c_t^{\delta}(1-n_t)^{1-\delta}]^{1-\gamma}}{1-\gamma}$ where n_t describes the hours of labor supplied in period t. In either case the production function is generalized to be of the form $f(k_t, n_t)e^{\tilde{\lambda}_t} = k_t^{\alpha}(n_t)^{1-\alpha}e^{\tilde{\lambda}_t}$.

The results for specification (1) are then compared with the Baseline cases of Table 4.4 while the specification (2) results are compared with those in Table 4.5. These results, and the intuition behind them, are easily summarized. First, if $\Omega = 1$, then the addition of a labor/leisure choice under either specification has no impact on the autocorrelations or ACTs for any of the financial series we study. If $\Omega < 1$, then the addition of a labor/leisure choice slightly diminishes the autocorrelations and ACTs for all the series.

To get an idea as to the magnitudes involved, see Table 4.7 where we present only the ACT measurements for comparison as we view them as more informative and intuitive.

Table 4.7

Panel A

$u(c_t) = \log c_t$ vs. $u(c_t, 1-n_t) = \log c_t + A\log(1-n_t)$

A.1 ACTs for $u(c_t) = \log c_t$

	$\Omega = 1$			$\Omega = 0.4$			$\Omega = 0.025$		
	$\rho = 0$	$\rho = .4$	$\rho = .95$	$\rho = 0$	$\rho = .4$	$\rho = .95$	$\rho = 0$	$\rho = .4$	$\rho = .95$
ACT(p_t^e)	2.696	3.827	13.89	3.77	5.49	19.92	11.31	16.89	74.63
ACT(p_t^b)	2.697	4.876	2.84	3.78	6.11	3.81	11.19	3.35	8.51
ACT(r_t^e)	1.658	1.968	2.61	1.83	2.32	3.58	2.01	2.737	8.33
ACT(r_t^b)	2.691	4.898	2.84	3.77	6.07	3.81	11.22	3.35	8.52
ACT(r_t^p)	2.02	2.02	2.02	2.02	2.02	2.02	2.02	2.02	2.02

A.2 ACTs for $u(c_t, 1-n_t) = \log c_t + A\log(1-n_t)$, A=2

	$\Omega = 1$			$\Omega = 0.4$			$\Omega = 0.025$		
	$\rho = 0$	$\rho = .4$	$\rho = .95$	$\rho = 0$	$\rho = .4$	$\rho = .95$	$\rho = 0$	$\rho = .4$	$\rho = .95$
ACT(p_t^e)	2.69	3.82	13.89	3.64	5.33	19.53	10.44	15.25	68.5
ACT(p_t^b)	2.69	4.88	2.84	3.66	6.17	3.71	10.52	3.39	8.05
ACT(r_t^e)	1.65	1.97	2.60	1.82	2.29	3.47	2.01	2.71	8.02
ACT(r_t^b)	2.69	4.89	2.84	3.66	6.4	3.73	10.53	3.39	8.07
ACT(r_t^p)	2.02	2.02	2.02	2.02	2.02	2.02	2.02	2.02	2.02

Panel B

$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$ vs. $u(c_t, 1-n_t) = \frac{\left[c_t^{\delta}(1-n_t)^{1-\delta}\right]^{1-\gamma}}{1-\gamma}$

B.1 ACTs for$(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$,

	$\Omega = 1$			$\Omega = .4$			$\Omega = .025$		
	$\rho = 0$	$\rho = .4$	$\rho = .95$	$\rho = 0$	$\rho = .4$	$\rho = .95$	$\rho = 0$	$\rho = .4$	$\rho = .95$
ACT(p_t^e)	3.60	5.25	19.38	5.40	8.16	30.49	21.79	31.16	129.88
ACT(p_t^b)	3.61	6.31	5.46	5.40	9.71	7.96	21.79	4.76	15.18
ACT(r_t^e)	2.11	2.74	3.91	2.28	3.15	6.31	2.09	2.99	13.72
ACT(r_t^b)	3.61	6.32	5.45	5.44	9.65	7.95	21.88	4.77	15.13
ACT(r_t^p)	2.02	2.02	2.02	2.02	2.02	2.02	2.02	2.02	2.02

B.2 ACTs for

$$u(c_t, 1-n_t) = \left[c_t^{\delta}(1-n_t)^{1-\delta}\right]^{1-\gamma}/1-\gamma\,, \delta = 0.5, \gamma = 5$$

	$\Omega = 1$			$\Omega = .4$			$\Omega = .025$		
	$\rho = 0$	$\rho = .4$	$\rho = .95$	$\rho = 0$	$\rho = .4$	$\rho = .95$	$\rho = 0$	$\rho = .4$	$\rho = .95$
ACT(p_t^e)	3.59	5.24	19.38	5.35	8.14	29.94	19.61	28.17	120.49
ACT(p_t^b)	3.59	6.29	5.48	5.37	9.65	7.91	19.38	4.49	13.49
ACT(r_t^e)	2.11	2.73	3.89	2.27	3.12	6.27	2.09	2.92	12.90
ACT(r_t^b)	3.60	6.27	5.45	5.37	9.59	7.29	19.42	4.50	13.57
ACT(r_t^p)	2.02	2.02	2.02	2.02	2.02	2.02	2.02	2.02	2.02

Once again, these results are not surprising. In the cases where $\Omega = 1$, the equilibrium level of hours provided, n_t, is independent of the shock and capital stock values. In the cases where $\Omega < 1$, the fact that the ACTs are all somewhat diminished indicates that the addition of a labor decision variable tends to pull the capital stock and consumption series back towards their means relative to an environment in which they are absent. This is a way of saying that variations in the supply of the agent's labor can be used to assist in stabilizing both the economy's capital stock series (and thus reduce the ACT of p_t^e) and its consumption series (and thus reduce the ACT of p_t^b), a fact well known in the business cycle literature. The effect is small, however, because the agent also prefers low variation in his leisure, $(1-n_t)$.

We close this section with a number of remarks:

1. For the representative agent class of models and its standard characterization mean aversion in equity and risk free return is the norm, except in the Baseline case where $\Omega = 1$, and shock correlation is low.

2. The various features that we have discussed generally enhance mean aversion in equity and risk free returns, at least as it is most commonly measured. These ACTs are generally larger.

3. The equity premium remains slightly mean averting for a very wide class of models we have studied.

In short, "mean aversion" appears to rule within this family of DSGE models.

6. Conclusion

We close by revisiting our objectives going forward. The results presented in Table 4.1 clearly demonstrate an absence of mean reversion – as it is conventionally defined (Property 1) for high persistence productivity disturbances.[22] It is also the case, however, that the elaborate constructs of Guvenen (2009) and Bansal and Yaron (2004) do yield mild mean reversion by Property I (near zero but negative autocorrelations in$\{\tilde{r}_t^e\}$ and $\{\tilde{r}_t^p\}$) even when persistence in the productivity shock is high. What accounts for the difference? It is to this topic that we turn in a companion paper. In particular, we explore a number of additional model features and assess their cumulative contributions in generating mean reversion in equity returns.[23]

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[22] From a business cycle perspective, a reasonable matching of the stylizsd facts demands high persistence in the productivity disturbance (see Cogley and Nason (1995)). It is in this sense that the high persistence cases are the only ones of empirical relevance.

[23] All the while we must restrict our explorations to models which simultaneously are able to replicate the most basic stylized business cycle facts.

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Technical Appendix

A. Proof of Proposition 2.1

a. Property II ➔ Property I

By Property II,

By Property II,

$$var(\tilde{x}_t + \tilde{x}_{t+1}) = var(\tilde{x}_t) + var(\tilde{x}_{t+1}) + 2cov(\tilde{x}_t, \tilde{x}_{t+1}) < 2var(\tilde{x}_t)$$

$$\rightleftharpoons var(\tilde{x}_{t+1}) + 2cov(\tilde{x}_t, \tilde{x}_{t+1}) < var(\tilde{x}_t).$$

Since $var(\tilde{x}_{t+1}) = var(\tilde{x}_t)$,

$$cov(\tilde{x}_t, \tilde{x}_{t+1}) < 0,$$

and Property I holds.

b. Property I together with the covariance condition (9) implies Property II. The proof is by induction.

Let j = 1, $var(\tilde{x}_t + \tilde{x}_{t+1}) = var(\tilde{x}_t) + var(\tilde{x}_{t+1}) + 2cov(\tilde{x}_t, \tilde{x}_{t+1}) < 2var(\tilde{x}_t)$ by Property I and the fact that $var(\tilde{x}_{t+1}) = var(\tilde{x}_t)$

Let j = 2,

$$var(\tilde{x}_t + \tilde{x}_{t+1} + \tilde{x}_{t+2}) = \sum_{j=0}^{2} var(\tilde{x}_{t+j}) + 2cov(\tilde{x}_t, \tilde{x}_{t+1}) + 2cov(\tilde{x}_t, \tilde{x}_{t+2})$$
$$+2cov(\tilde{x}_{t+1}, \tilde{x}_{t+2})$$

by Property I

$2cov(\tilde{x}_t, \tilde{x}_{t+1}) < 0$ and by condition (9),

$$2cov(\tilde{x}_t, \tilde{x}_{t+1}) + 2cov(\tilde{x}_t, \tilde{x}_{t+2}) < 0$$

by Property I, $2cov(\tilde{x}_{t+1}, \tilde{x}_{t+2}) < 0$. Therefore,

$$var(\tilde{x}_t + \tilde{x}_{t+1} + \tilde{x}_{t+2}) = \sum_{j=0}^{2} var(\tilde{x}_{t+j}) = 3var(\tilde{x}_t)$$

Suppose, by Property I and condition (9), that

$$var(\tilde{x}_t + \tilde{x}_{t+1} + \cdots + \tilde{x}_{t+j-1}) < jvar(\tilde{x}_t).$$

To show $var(\tilde{x}_t + \tilde{x}_{t+1} + \cdots + \tilde{x}_{t+j}) < (j+1)var(\tilde{x}_t)$.

$$var(\tilde{x}_t + \tilde{x}_{t+1} + \cdots + \tilde{x}_{t+j}) = var(\tilde{x}_t + \tilde{x}_{t+1} + \cdots + \tilde{x}_{t+j-1}) + var(\tilde{x}_{t+j})$$
$$+2\sum_{s=0}^{j-2} cov(\tilde{x}_{t+s}, \tilde{x}_{t+j}) + 2cov(\tilde{x}_{t+j-1}, \tilde{x}_{t+j})$$

$$< jvar(\tilde{x}_t) + jvar(\tilde{x}_{t+1}) + 2\sum_{s=0}^{j-2} cov(\tilde{x}_{t+s}, \tilde{x}_{t+j}) + 2cov(\tilde{x}_{t+j-1}, \tilde{x}_{t+j})$$

(by induction)

by Property I, $cov(\tilde{x}_{t+j-1}, \tilde{x}_{t+j}) < 0$.

Thus by condition (9),

$$2\sum_{s=0}^{j-2} cov(\tilde{x}_{t+s}, \tilde{x}_{t+j}) + 2cov(\tilde{x}_{t+j-1}, \tilde{x}_{t+j}) < 0 \quad .$$

Therefore, since $var(\tilde{x}_{t+j}) = var(\tilde{x}_t)$

$var(\tilde{x}_t + \tilde{x}_{t+1} + \cdots + \tilde{x}_{t+j}) < (j+1)var(\tilde{x}_t)$.

Basically, for I ➔ II, the persistence of the series must rapidly decline.

B. Proposition 3.1 part b, mean aversion in dividends.

By (15),

$$\begin{aligned}
cov(\tilde{d}_t, \tilde{d}_{t+1}) &= cov(\alpha(1-\beta)\tilde{k}_t^\alpha, \alpha(1-\beta)\tilde{k}_{t+1}^\alpha \tilde{\lambda}_{t+1}) \\
&= (\alpha(1-\beta))^2 cov\left(\tilde{k}_t^\alpha, \tilde{\lambda}_t\left(\alpha\beta\tilde{k}_t^\alpha\tilde{\lambda}_t\right)^\alpha \tilde{\lambda}_{t+1}\right) \\
&= (\alpha(1-\beta))^2 (\alpha\beta)^2 cov\left(\tilde{k}_t^\alpha, \tilde{\lambda}_t\left(\tilde{k}_t^\alpha\tilde{\lambda}_t\tilde{\lambda}_{t+1}\right)^\alpha\right) \\
&= (\alpha(1-\beta))^2 (\alpha\beta)^2 \left\{E\left(\tilde{k}_t^{\alpha+\alpha^2}\tilde{\lambda}_t^{1+\alpha}\tilde{\lambda}_{t+1}\right) - E\left(\tilde{k}_t^\alpha\tilde{\lambda}_t\right)E\left(\tilde{k}_t^{\alpha^2}\tilde{\lambda}_t^\alpha\tilde{\lambda}_{t+1}\right)\right\} \\
&= (\alpha(1-\beta))^2 (\alpha\beta)^2 E(\tilde{\lambda})\left\{E\left(\tilde{k}_t^{\alpha+\alpha^2}\right)E\left(\tilde{\lambda}_t^{1+\alpha}\right) - E(\tilde{k}_t^\alpha)E(\tilde{\lambda}_t)E\left(\tilde{k}_t^{\alpha^2}\right)E(\tilde{\lambda}_t^\alpha)\right\} \\
&\geq 0
\end{aligned}$$

since, by the Harris (1960), inequality

$$E\left(\tilde{k}_t^{\alpha+\alpha^2}\right) \geq E(\tilde{k}_t^\alpha)E\left(\tilde{k}_t^{\alpha^2}\right),$$

and $$E\left(\tilde{\lambda}_t^{1+\alpha}\right) \geq E(\tilde{\lambda}_t)E(\tilde{\lambda}_t^\alpha).$$

C. Derivation of risk free bond price.

$$\begin{aligned}
p_t^b &= \beta \int \frac{u_1(c_{t+1})}{u_1(c_t)} dF(\tilde{c}_{t+1}, c_t) \\
&= \beta \int \frac{(1-\alpha\beta)k_t^\alpha\lambda_t}{(1-\alpha\beta)[\alpha\beta k_t^\alpha\lambda_t]^\alpha\tilde{\lambda}_{t+1}} dF(\tilde{\lambda}_{t+1}) \\
&= \beta \int \frac{1}{(\alpha\beta)^\alpha[k_t^\alpha\lambda_t]^{\alpha-1}\tilde{\lambda}_{t+1}} dF(\tilde{\lambda}_{t+1}) \\
&= \frac{\beta}{(\alpha\beta)^\alpha} k_t^{\alpha(1-\alpha)}\lambda_t^{1-\alpha}E(\lambda^{-1}), \text{ since } \tilde{\lambda}_t \text{ is i.i.d.}
\end{aligned}$$

Thus, $p_t^b = \frac{\beta E(\lambda^{-1})}{(\alpha\beta)^\alpha} k_t^{\alpha(1-\alpha)} \lambda_t^{1-\alpha}$.

D. Proof of Proposition 3.2

$cov(\tilde{k}_t, \tilde{k}_{t+1})$

$$= cov(\tilde{k}_t, i(\tilde{k}_t, \tilde{\lambda}_t) + (1-\Omega)\tilde{k}_t)$$

$$= \iint (\tilde{k}_t - \bar{k})(i(\tilde{k}_t, \tilde{\lambda}_t) + (1-\Omega)\tilde{k}_t - \bar{k}) dF(\tilde{k}_t) dG(\tilde{\lambda}_t)$$

Let $f_1(\tilde{k}_t, \tilde{\lambda}_t) = \tilde{k}_t - \bar{k}$ and $f_2(\tilde{k}_t, \tilde{\lambda}_t) = i(\tilde{k}_t, \tilde{\lambda}_t) + (1-\Omega)\tilde{k}_t - \bar{k}$. Both $f_1(\ )$ and $f_2(\ )$ are increasing functions of their arguments by assumption.

By FKG ( ) or Harris (1960),

$$= \iint (\tilde{k}_t - \bar{k})(i(\tilde{k}_t, \tilde{\lambda}_t) + (1-\Omega)\tilde{k}_t - \bar{k}) dF(\tilde{k}_t) dG(\tilde{\lambda}_t) \geq \iint (\tilde{k}_t - \bar{k}) dF(\tilde{k}_t) dG(\tilde{\lambda}_t)$$

$$\times \iint (i(\tilde{k}_t, \tilde{\lambda}_t) + (1-\Omega)\tilde{k}_t - \bar{k}) dF(\tilde{k}_t) dG(\tilde{\lambda}_t) = 0$$

Thus, $cov(\tilde{p}_t^e, \tilde{p}_{t+1}^e) = cov(\tilde{k}_t, \tilde{k}_{t+1}) \geq 0$

.

In the case of p_t^b,

$$cov(\tilde{p}_t^b, \tilde{p}_{t+1}^b) = cov\left(h(\tilde{k}_t, \tilde{\lambda}_t), h(\tilde{k}_{t+1}, \tilde{\lambda}_{t+1})\right)$$

$$= cov\left(h(\tilde{k}_t, \tilde{\lambda}_t), h(i(\tilde{k}_t, \tilde{\lambda}_t) + (1-\Omega)\tilde{k}_t, \tilde{\lambda}_{t+1})\right)$$

and continue along the same lines as the proof above since $h(i(\tilde{k}_t, \tilde{\lambda}_t) + (1-\Omega)\tilde{k}_t, \tilde{\lambda}_{t+1})$ is an increasing function of all its arguments.

E. Follow on to Proposition 3.3: Proof that $cov(\tilde{r}_t^b, \tilde{r}_{t+1}^b) \leq 0$.

$$cov(\tilde{r}_t^b, \tilde{r}_{t+1}^b) = cov\left(\frac{(\alpha\beta)^2}{\beta E(\lambda^{-1})} k_t^{\alpha(\alpha-1)} \lambda_t^{\alpha-1}, \frac{(\alpha\beta)^2}{\beta E(\lambda^{-1})} k_{t+1}^{\alpha(\alpha-1)} \lambda_{t+1}^{\alpha-1}\right)$$

$$= \left(\frac{(\alpha\beta)^2}{\beta E(\lambda^{-1})}\right)^2 cov\left(k_t^{\alpha(\alpha-1)} \lambda_t^{\alpha-1}, (\alpha\beta k_t^\alpha \lambda_t)^{\alpha(\alpha-1)} \lambda_{t+1}^{\alpha-1}\right)$$

$$= (\alpha\beta)^{\alpha(\alpha-1)} \left(\frac{(\alpha\beta)^2}{\beta E(\lambda^{-1})}\right)^2 cov\left(\tilde{k}_t^{\alpha^2-\alpha} \tilde{\lambda}_t^{\alpha-1}, \tilde{k}_t^{\alpha^3-\alpha^2} \tilde{\lambda}_t^{\alpha^2-\alpha} \tilde{\lambda}_{t+1}^{\alpha-1}\right)$$

By the properties of the covariance function and that k_t, λ_t and λ_{t+1}

are all independent of one another, the RHS expression becomes

$$= ME\left(\tilde{\lambda}_t^{\alpha-1}\right)\left\{E\left(\tilde{k}_t^{\alpha^3-\alpha}\right)E\left(\tilde{\lambda}_t^{\alpha^2-1}\right) - E\left(\tilde{k}_t^{\alpha^2-\alpha}\right)E\left(\tilde{\lambda}_t^{\alpha-1}\right)E\left(\tilde{k}_t^{\alpha^3-\alpha^2}\right)E\left(\tilde{\lambda}_t^{\alpha^2-\alpha}\right)\right\}.$$

Let $f_1(k) = k_t^{\alpha^2-\alpha}$ and $f_2(k) = k^{\alpha^3-\alpha^2}$; each is a decreasing function of *k*, furthermore, $f_1(k)f_2(k) = k^{\alpha^3-\alpha}$which is also decreasing in *k*. By FKG or Harris (1960), j

$$E\left(\tilde{k}_t^{\alpha^3-\alpha}\right) \geq E\left(\tilde{k}_t^{\alpha^2-\alpha}\right)E\left(\tilde{k}_t^{\alpha^3-\alpha^2}\right).$$

and,

$$E\left(\tilde{\lambda}_t^{\alpha^2-1}\right) \geq E\left(\tilde{\lambda}_t^{\alpha-1}\right)E\left(\tilde{\lambda}_t^{\alpha^2-\alpha}\right).$$

Thus

$cov\left(\tilde{r}_t^b, \tilde{r}_{t+1}^b\right) \geq 0.$ We have been unable to derive any definitive result for the premium.

F. Proof of Proposition 4.3

Knowing that for $y \sim N(0, V), E[exp(y)] = \frac{V}{2}$ and for the *AR*(1) process, $Var(x_t) = \sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)$:

$$\begin{aligned}
Cov(\lambda_t, \lambda_{t+1}) &= Cov\big(exp(x_t), exp(\rho x_t + \varepsilon_{t+1})\big) \\
&= E[exp(x_t), exp(\rho x_t + \varepsilon_{t+1})] - E[exp(x_t)]E[exp(\rho x_t + \varepsilon_{t+1})] \\
&= E\big[exp\big((\rho+1)x_t + \varepsilon_{t+1}\big)\big] - E[exp(x_t)]E[exp(\rho x_t + \varepsilon_{t+1})] \\
&= E\big[exp\big((\rho+1)x_t\big)\big]E[exp(\varepsilon_{t+1})] \\
&\qquad - E[exp(x_t)]E[exp(\rho x_t)]E[exp(\varepsilon_{t+1})] \\
&= E[exp(\varepsilon_{t+1})]\big(E\big[exp\big((\rho+1)x_t\big)\big] - E[exp(x_t)]E[exp(\rho x_t)]\big) \\
&= exp\left(\frac{\sigma^2}{2}\right)\left(exp\left(\frac{1}{2}(\rho+1)^2\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)\right. \\
&\qquad \left. - exp\left(\frac{1}{2}\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)exp\left(\frac{1}{2}\rho^2\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)\right)
\end{aligned}$$

$$= exp\left(\frac{\sigma^2}{2}\right)\left(exp\left(\frac{1}{2}(\rho^2 + 2\rho + 1)\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right) - exp\left(\frac{1}{2}(\rho^2 + 1)\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)\right)$$

$$= exp\left(\frac{\sigma^2}{2}\right)\left(exp\left(\frac{1}{2}(\rho^2 + 1)\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right) - exp\left(\frac{1}{2}(\rho^2 + 1)\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)\right)$$

$$= exp\left(\frac{\sigma^2}{2} + \frac{1}{2}(\rho^2 + 1)\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)\left(exp\left(\rho\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right) - 1\right)$$

$$= exp\left(\frac{\sigma^2}{2}\left(1 + (\rho^2 + 1)\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)\right)\left(exp\left(\rho\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right) - 1\right)$$

Thus $Cov(\lambda_t, \lambda_{t+1}) = exp\left(\frac{\sigma^2}{2}\left(1 + (\rho^2 + 1)\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right)\right)\left(exp\left(\rho\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right)\right) - 1\right)$.

Now, clearly, the first element is positive since it is an exponent. Further, since

$$\sigma^2\left(\frac{1-\rho^{2t}}{1-\rho^2}\right) > 0,$$

we know that the second element is a strictly increasing function of ρ, reaching a value of zero at $\rho = 0$. Therefore, for $\rho < 0$, the expression is negative, while for $\rho > 0$ the expression is positive.